Plan
1. Statement and the strategy of the proof
2. Mochizuki's formula and its proof
3. Instanton partition function
4. Seiberg-Witten curve
5. Final computation

1. Statement

X : C^∞ 4-mfd $\quad b^+ > 1$, $b_1 = 0$

$(K_X^2) := 2\chi(X) + 3\sigma(X) \qquad \chi_h(X) := \dfrac{\chi(X)+\sigma(X)}{4} \qquad$ as if X : cpx surface

▷ Donaldson invariants

$$\mathcal{D}^{\xi}(\exp(\alpha z + p x)) := \sum_y \Lambda^{\dim M(y)} \int_{M^U(y)} \exp(\mu(\alpha z + p x))$$

z, x : variable $\quad \alpha \in H_2(X)$, $p = pt \in H_0(X)$

$y = (2, \xi, n) \in H^*(X) \qquad \xi$: fix , but move n

$M(y)$: moduli of $U(2)$-instanton with Chern classes $= y$

$M^U(y)$: Uhlenbeck compactification

$= M(y) \sqcup X \times M(y-(0,0,1)) \sqcup S^2X \times M(y-(0,0,2)) \cup \cdots$

$\mathcal{E}$: universal b'dle on $X \times M(y)$

$$\mu(\alpha z + p x) = \int_X \left(c_2(\mathcal{E}) - \tfrac{1}{4} c_1(\mathcal{E})^2\right) \cup (\alpha z + p x) \in H^*(M^U(y))$$

Rem X : proj. surface$/\mathbb{C}$ $\quad H$: ample line b'dle

$\Rightarrow$ We can use $M_H(y)$ = moduli space of semistable torsion-free sheaves

instead of $M^U(y)$. (independent of H)

Rem. $M^U(y) \cong M^U(y \cdot e^{c_1(L)})$ $\quad L$: line b'dle

$\Rightarrow \mathcal{D}^{\mathfrak{z}} = \mathcal{D}^{\mathfrak{z}+2c_1(L)}$ up to sign

Def. X : KM-simple type $\overset{\text{def.}}{\iff} \left(\frac{\partial^2}{\partial x^2} - 4\Lambda^2\right)\mathcal{D}^{\mathfrak{z}} = 0$

The meaning of this condition is difficult to understand.
But all known examples of 4-mfds (with $b^+ > 1$) satisfy this conditions.

▷ Seiberg-Witten invariants

$\mathfrak{s}$: spin^c str. $\quad c_1(\mathfrak{s}) = c_1(S^+) \rightsquigarrow SW(\mathfrak{s})$: SW invariant $\in \mathbb{Z}$

$\begin{cases} \phi : \text{spinor} \\ A : \text{spin connection} \end{cases} \qquad \begin{cases} \not{D}_A \phi = 0 \\ F_A^+ = \mu(\phi) \end{cases}$

<u>Rem</u>. X : proj. surface $/\mathbb{C}$ $\qquad c_1(\mathfrak{s}) = c_1(2\xi - K_X) \qquad \xi \in NS(X)$

SW moduli space $= \{\text{rk } 1 \text{ stable pairs}\}/\text{iso.}$

$= \{\text{line b'dles } L + \text{sections } s \in H^0(L) \mid c_1(L) = \xi\}/\text{iso.}$

(Need: virtual counting) $\quad$ or $\{(L, s \in H^0(K_X - L)) \mid c_1(L) = \xi\}/\text{iso.}$

← easy to compute!!

<u>Def.</u> X : SW simple type $\Leftrightarrow$ SW inv. $\neq 0$ only if

v.dim. of moduli sp. $= 0 \quad \underset{\text{def.}}{\Leftrightarrow} c_1(\mathfrak{s})^2 = (K_X^2)$

No non-simple type 4-mfd with $b^+ \geq 3$ is found so far.

<u>Witten's conjecture (1994)</u>

X : SW simple type

$\Longrightarrow$ KM simple type &

$$\mathcal{D}^{\xi}(\exp \alpha \times (1 + \tfrac{1}{2}p)) = 2^{(K_X^2) - \chi_h(X) + 2} (-1)^{\chi_h(X)} e^{(\alpha^2)/2}$$

$$\times \sum_{\mathfrak{s}} SW(\mathfrak{s}) (-1)^{(\xi, \xi + c_1(\mathfrak{s}))/2} e^{(c_1(\mathfrak{s}), \alpha)} \quad \text{(finite sum)}$$

Main Thm. [Göttsche-N-Yoshioka]

Conjecture is true for a cpx proj. surface X

Strategy of the proof (rough: need a modification)

1. [Mochizuki]

   Relate $\mathcal{D}$ and $SW(\mathfrak{s})$ via moduli spaces of rk 2 stable pairs.

   $\longrightarrow$ Similar formula as above, but coeff's are given integrals over Hilbert schemes of pts on X

2. [Ellingsrud-Göttsche-Lehn]

   The above integrals are universal, i.e. can be determined by the answers for X = toric surface ← I will skip this part!

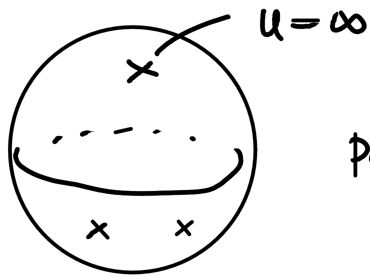
3. fixed pt formula $\longrightarrow$ reduce to $X = \mathbb{C}^2$

   then answers are given in terms of instanton partition function

4. Compute instanton partition function in terms of elliptic integral (elliptic curves = Seiberg-Witten curves)

5. Apply residue thm :

Need to compute $\operatorname{Res}_{u=\infty}(\cdots)$ ← explicit, but very complicated



parameter sp. for elliptic curves

Compute residues at other poles ⟶ easy

This is close to Witten's physical intuition.

<u>Rem</u>. Feehan–Leness (based on an earlier idea by Pidstrigach–Tyurin) proved more general class of 4-mfds under some technical assumption on a property of $SO(3)$-monopole moduli spaces

The step 1 in above strategy + indirect arguments

Mochizuki's work was motivated by [FL, PT], but is based on
- virtual fundamental classes and
- virtual localization

⟶ efficient for computation